Geometric Transformations of the Plane

Lecture 6   Feb 14, 2021

Description of the idea

- **Idea.** We will study basic **moves** that change shapes in plane
- We learn how these moves change basic shapes such as a line or circle, or quantities such as length and angle
- Finally, we combine these moves to make more complicated moves

We call these moves geometric transformations

Examples

(1) **Translation**: We move every point in certain direction, with certain distance.

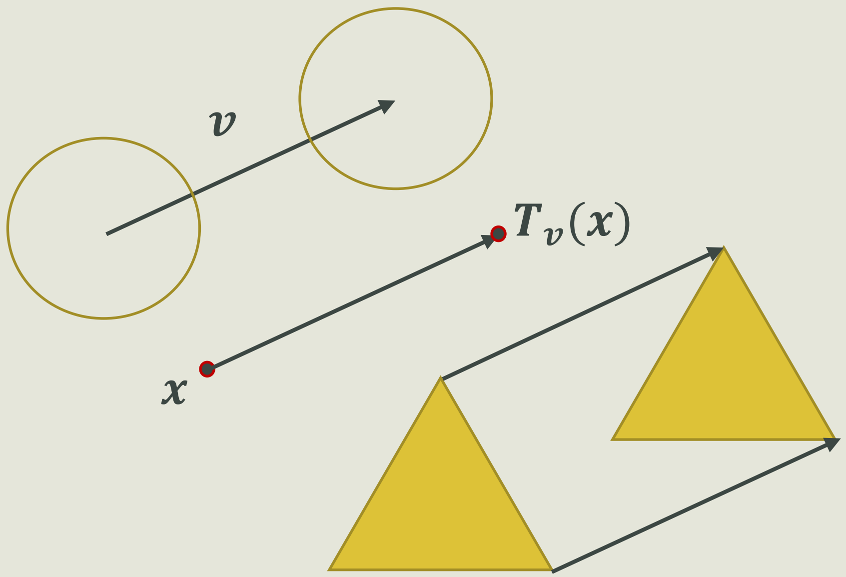
- A translation $\boldsymbol{T_v}$ is described by a **vector $\boldsymbol{v}$:**

$\boldsymbol{x} \rightarrow \boldsymbol{T_v(x)} = \boldsymbol{x} + \boldsymbol{v}$

- Translations **preserve** length and angle

- It does not fixes any point unless $\boldsymbol{v}$ = 0

- If $\boldsymbol{v} = 0$, then T does not make any change

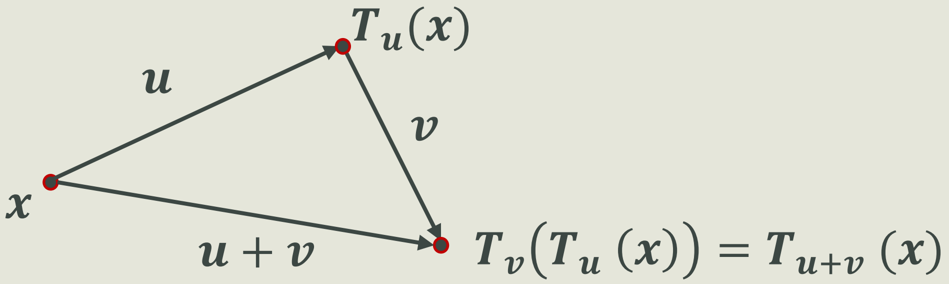


Doing multiple moves

- **Description:** If you first move points with a vector u and then with a vector v, it is as if you have moved points with sum of two vectors $u + v$

- Mathematical notation: $T_v \circ T_u = T_{u+v}$

WE SAY : <u>Composition</u> of a series of translations by vectors $v_1, \dots, v_k$ is a translation by the vector $v = v_1 + \cdots + v_k$



Examples

(2) Homothecy (or Homothety): A transformation of plane which dilates distances with respect to a fixed point (center of Homothecy)

- A Homothecy $\boldsymbol{H_{o,r}}$ is described by its center point

o and the dilation factor r:

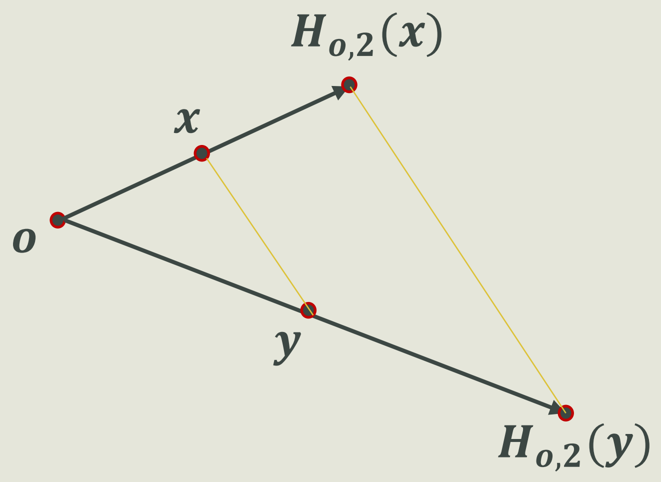
$$\boldsymbol{x} \;\rightarrow\; \boldsymbol{H_{o,r}(x) = o + r\,(x - o)}$$

- Homothecy **preserve** angle but

  changes length by a factor of r

- It only fixes the center point o (unless $r = 1$)

- If $\boldsymbol{r} = 1$, then H does not make any change



More pictures

- $r > 1$: things get bigger. $r < 1$: things get smaller
- Effect of Homothecy on

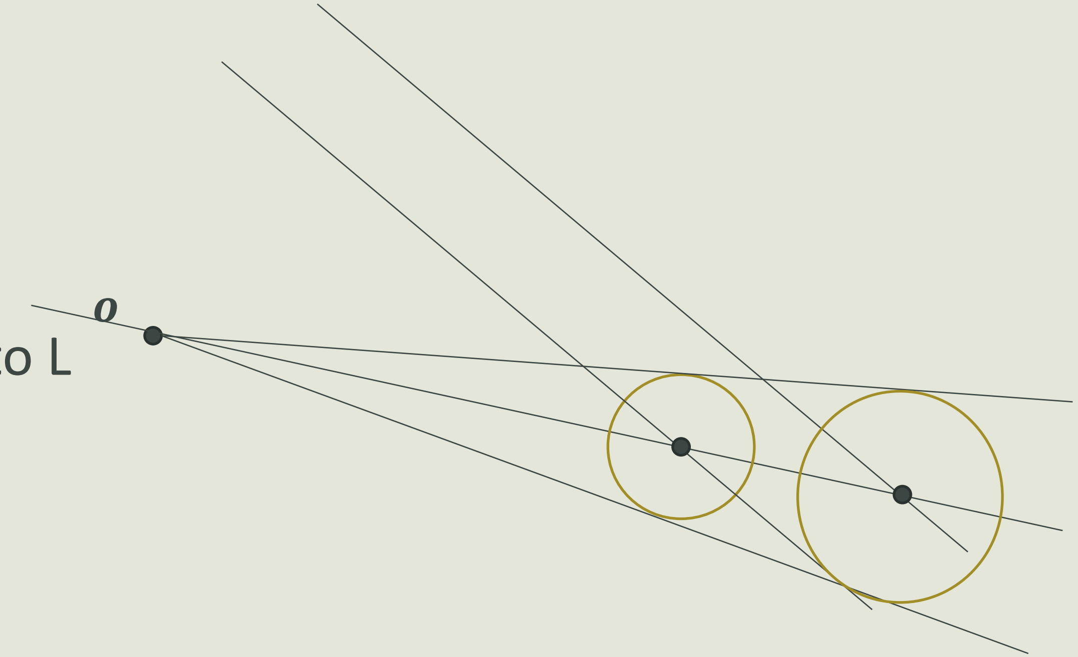
Lines:

(1) every line L will be mapped to another line parallel to L

(2) If L passes through the center it will be preserved

Circles:

every Circle C will be mapped to another circle



Composition of two Homothecies

- If we first do a Homothecy $H_1 = H_{o_1,r_1}$ with center o_1 and dilation factor r_1, and then do another a Homothecy $H_2 = H_{o_2,r_2}$ with center o_2 and dilation factor r_2, what is the over all transformation?
- **Mathematical notation**: What can we say about $H = H_2 \circ H_1$?
- Observations:
- (1) H does not change angles.
- (2) H changes distances by a total factor of $r = r_1 r_2$

(Q) Could H possibly be a Homothecy as well with a dilation factor of r

(Q) If yes, where will be its center?

Where is the center of H?

- **Lemma:** If $r_1 r_2 = 1$, then the composition of H_1 and H_2 is a translation

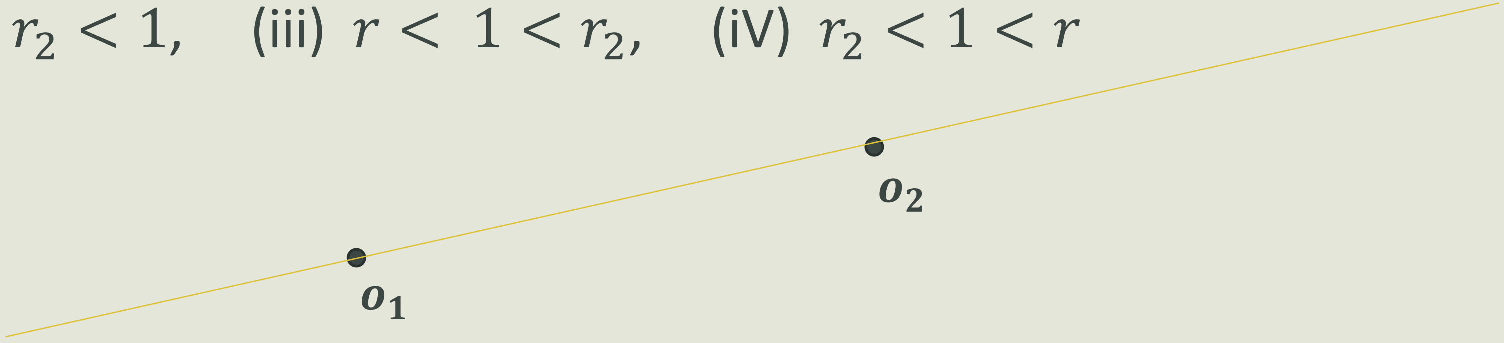
by the vector $v = r_2 \ \overrightarrow{o_1 o_2}$

- **Lemma:** If $r_1 r_2 \neq 1$, then the composition of H_1 and H_2 is a homothety with ratio $r = r_1 r_2$

and the center o will be somewhere on the line $o_1 o_2$

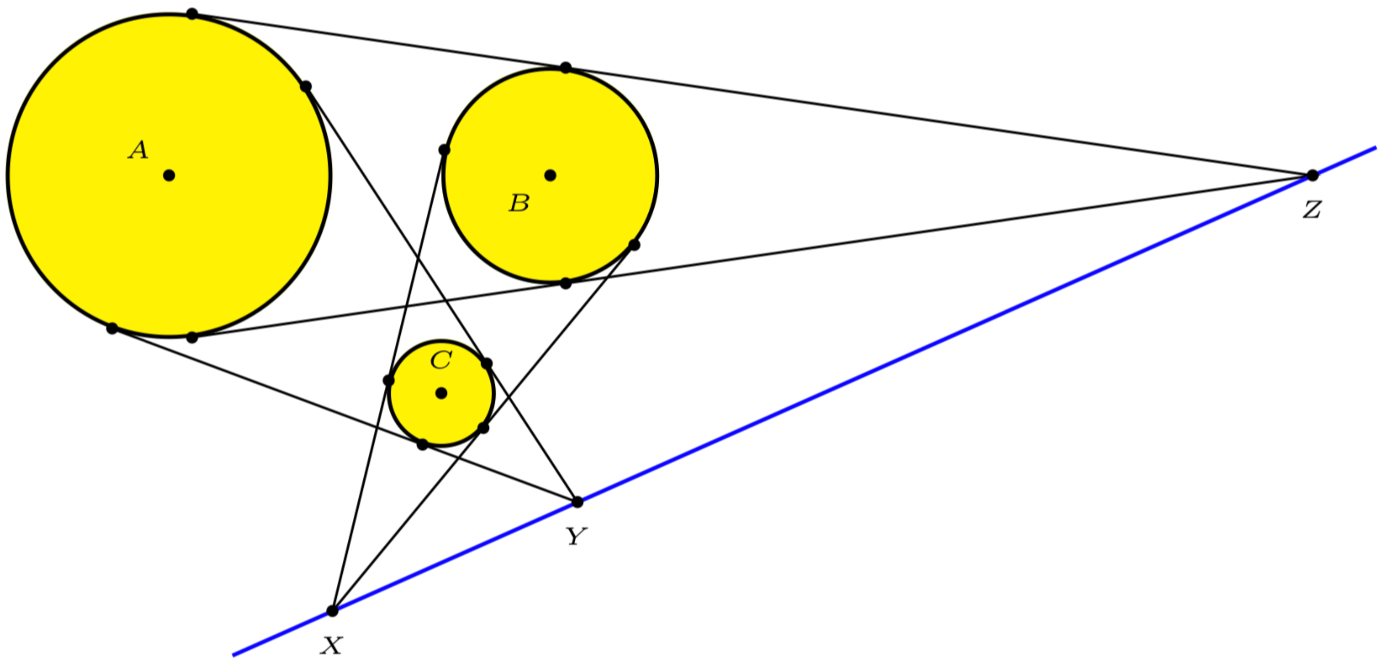
- Four cases:

(i) $r, r_2 > 1$, (ii) $r, r_2 < 1$, (iii) $r < 1 < r_2$, (iV) $r_2 < 1 < r$



Lets rework a problem form last time

4. Given three circles with centers A, B, C and distinct radii, show that the exsimilicenters of the three pairs of circles are collinear.



Strategy of the proof

- 1. We perform 3 homotheties with centers x, y , z and ratios $\frac{r_B}{r_C}, \frac{r_A}{r_B}, and \frac{r_C}{r_A}$ respectively.
- 2. The circle C is fixed in this process
- 3. The overall ration is 1.
- 4. So the composition of these 3 homotheties does not move any point

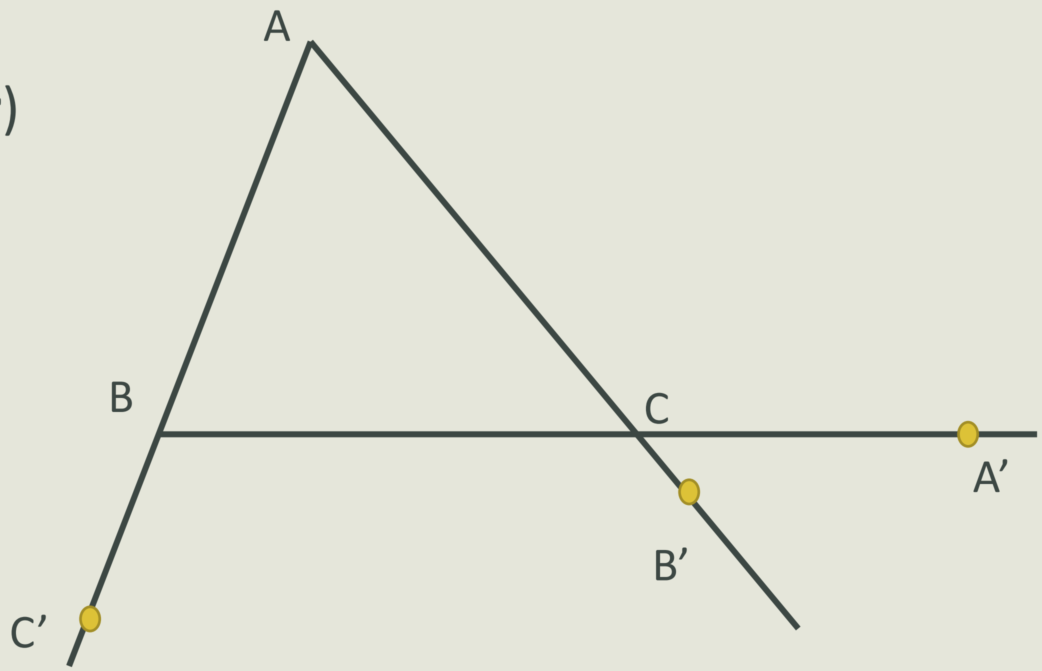
Now

- 5. Track the line passing through xy in this process to conclude that z must be on it as well.

Proof of Menelaus (version 2) with homothety

Theorem. Suppose $\frac{B'A}{B'C} \times \frac{A'C}{A'B} \times \frac{C'B}{C'A} = 1$ then

A’, B’, C’ are on one line L (we say A’, B’, C’ are **collinear**)



Repeat the same proof as the previous one.

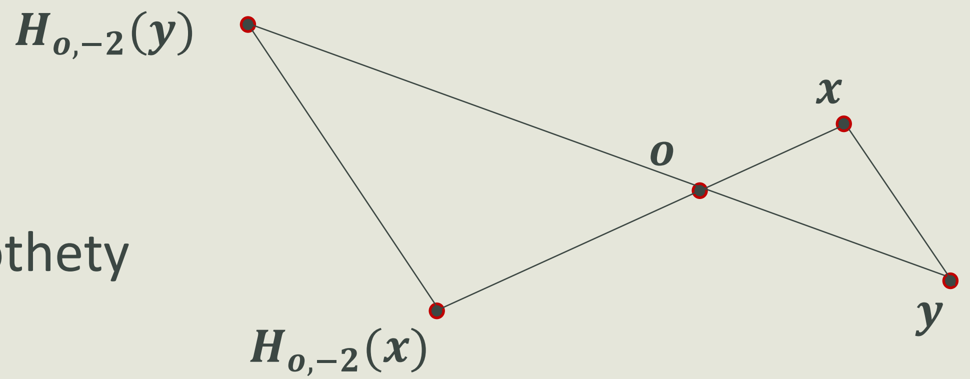
Problems to solve with Homothecy

- https://www.cut-the-knot.org/pythagoras/Transforms/ProblemsByHomothety.shtml
- https://eldorado.tu-dortmund.de/bitstream/2003/31745/1/195.pdf
- http://users.math.uoc.gr/~pamfilos/eGallery/problems/Similarities.pdf

Homothety with negative scaling factor

We can also consider Homotheties $\boldsymbol{H}_{(\boldsymbol{o},\boldsymbol{r})}$ where the dilation factor $r < 0$:

$$\boldsymbol{x} \rightarrow \boldsymbol{H}_{\boldsymbol{o},\boldsymbol{r}}(\boldsymbol{x}) = \boldsymbol{o} + \boldsymbol{r}\,(\boldsymbol{x} - \boldsymbol{o})$$



The same rule as before applies:

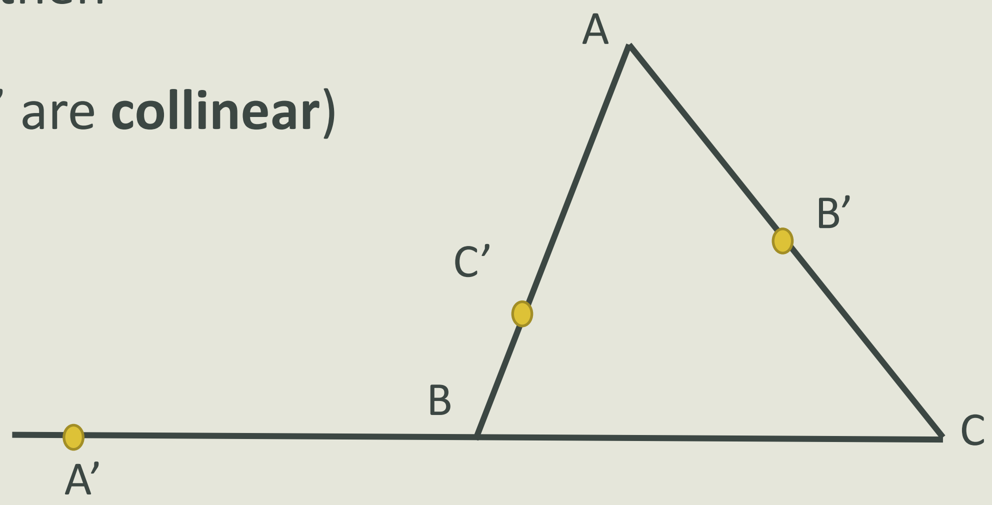
Composition of Homotheties is a homothety

(sometimes a translation)

Proof of Menelaus (version 1) with Homothety

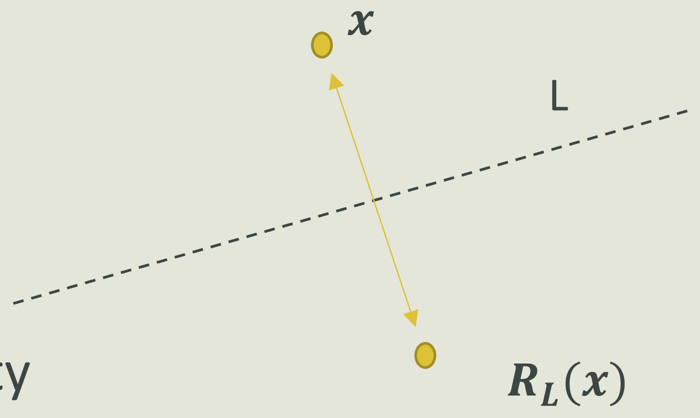
Theorem. Suppose $\frac{B'A}{B'C} \times \frac{A'C}{A'B} \times \frac{C'B}{C'A} = 1$ then

A’, B’, C’ are on one line L (we say A’, B’, C’ are **collinear**)



More geometric transformations

- (3) **Reflection** across a line L:



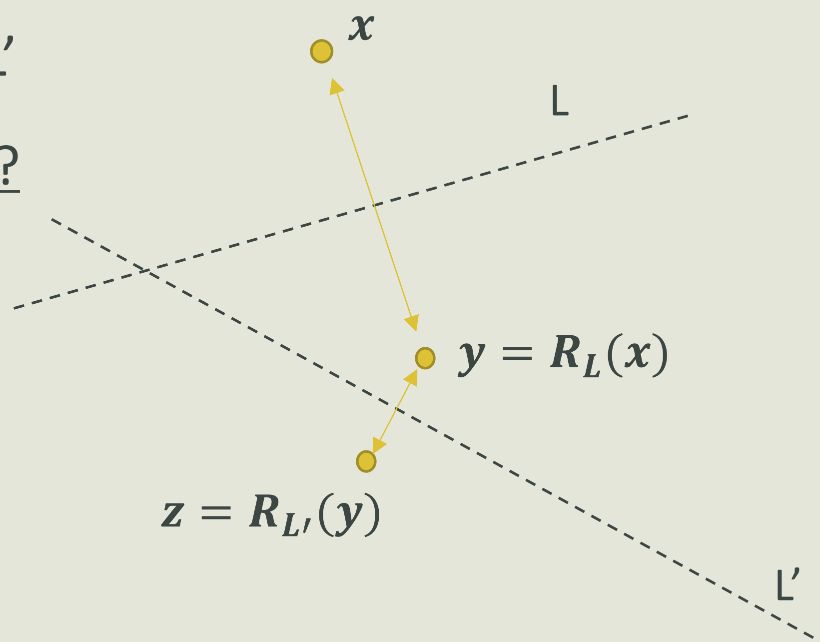
- Properties:

- Applying R_L twice, we get identity
- It preserves distance and angle

Composition of two reflections

- (3) If we reflect across a line L and then L'

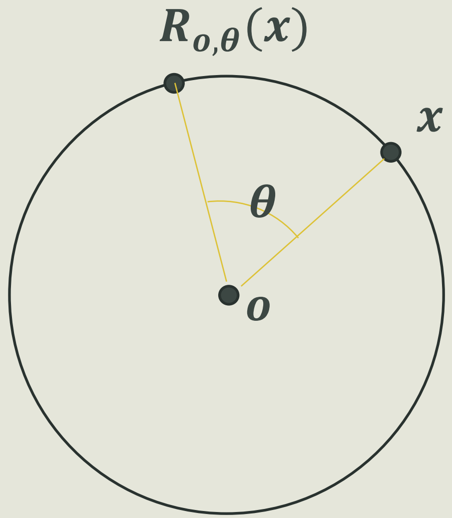
<u>what is the composition of two reflections?</u>



More geometric transformations

- (4) **Rotation** around a point o with angle θ
- Properties:

- It preserves distance and angle
- The only point fixed is the center o

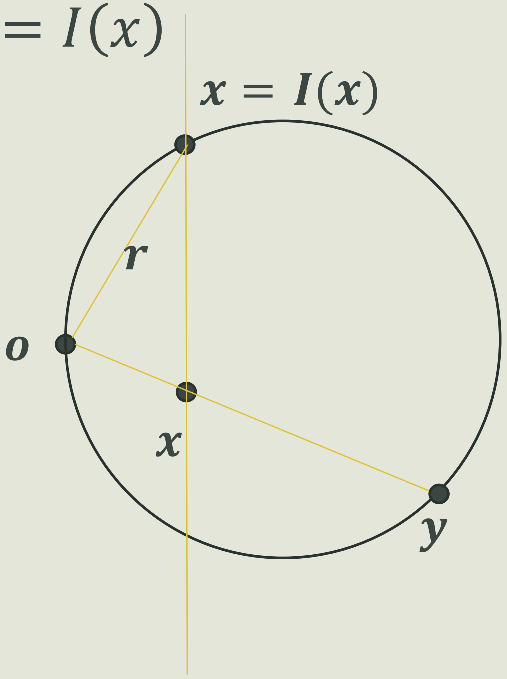


More geometric transformations

- (5) **Inversion** with respect to a point o and product distance r^2 :

It takes any point x besides the center o to another point $y = I(x)$

Colinear with x and o such that $ox \times oy = r^2$



It changes any line not passing through the center o to a circle (as in the picture). In this regard, it is different from transformations we had before.